

A radiation-like era before inflation

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Abstract

We show that the semiclassical approximation to the Wheeler-DeWitt equation for the minisuperspace of a minimally coupled scalar field in the spatially flat de Sitter Universe prompts the existence of an initial power-law evolution driven by non-adiabatic terms from the gravitational wavefunction which act like radiation. This simple model hence describes the onset of inflation from a previous radiation-like expansion during which the cosmological constant is already present but subleading.

1 Introduction

The inflationary paradigm is a basic ingredient of the standard model of cosmology which is well-known for solving several problems of the old big bang theory [1]. Many different mechanisms have been proposed to describe both the onset of such a phase and how it ends, most of which rely on the existence of an “inflaton” (scalar) field. The stress tensor of the latter plays the role of an effective cosmological constant and drives the accelerated expansion of the universe through the Einstein equations, but its nature – whether fundamental or effective – is not clearly understood yet. It is therefore worth considering alternative mechanisms to realise such a scenario.

We shall here address this problem by employing the Born-Oppenheimer approach in the semiclassical approximation [2] for the Wheeler-DeWitt equation [3] in the minisuperspace of the cosmic scale factor a and one mode of a minimally coupled scalar field ϕ , the latter representing matter and not the inflaton. Our approach closely follows that of Refs. [2, 4] and leads to two equations governing, respectively, the gravitational wavefunction $\psi = \psi(a)$ and the matter quantum state $\chi = \chi(\phi)$. It was already shown in Ref. [5] that quantum gravitational fluctuations become negligible for “large” a (late times) and, if a cosmological constant Λ is present, the de Sitter evolution (that is, eternal inflation) is recovered. The opposite regime of “small” a (early times) was investigated in Ref. [6], in which we showed that quantum gravitational fluctuations are again negligible and matter states χ become asymptotically free on approaching the limit of the semiclassical regime.

Although the consistency of such a matter behaviour with all the approximations employed was checked *a posteriori*, possible corrections to the evolution of a were not considered in Ref. [6]. In the following, we shall therefore analyse in details the equation governing a for asymptotically free matter states and show that the terms coming from the WKB approximation for the gravitational wavefunction ψ , which would vanish in the adiabatic approximation, actually dominate the evolution of the scale factor at very early times (like the cases treated in Ref. [7]). In particular, we shall uniquely identify

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one consistent coupled dynamics for matter and gravity in which a evolves like in a radiation-dominated universe, although no radiation is actually present. In this scenario, the onset of inflation occurs at the end of the semiclassical regime without the need of a dynamical inflaton field.

In Section 2, we briefly review the general formalism of Ref. [5] and the relevant results from Ref. [6]. The equation for gravity will then be analysed in the very early regime in Section 3 and the equation for matter in Section 4, where we uniquely identify the radiation-like behaviour for a . We finally comment and compare with previous results in Section 5.

We use units with $\hbar = c = 1$.

2 Minisuperspace semiclassical cosmology

The Wheeler-DeWitt equation in the minisuperspace of one mode of wavenumber k of a minimally coupled scalar field $\phi = \phi(\tau)$ and the scale factor $a = a(\tau)$, can be written as

$$\hat{H} \Psi(a, \phi) = \left(\hat{H}_G + \hat{H}_\phi \right) \Psi(a, \phi) = 0 , \quad (2.1)$$

where H_G and H_ϕ are the Hamiltonians for a and ϕ respectively [2, 4, 5]. From the Born-Oppenheimer decomposition of the universe wavefunction,

$$\Psi(a, \phi) = \psi(a) \chi(\phi; a) , \quad (2.2)$$

after several manipulations and redefinitions, one obtains an equation for the gravitational part (see Eq. (14b) of Ref. [5] with tildes omitted for simplicity),

$$\left(\frac{\ell_p^2}{2a} \partial_a^2 + \frac{a^3 \Lambda}{6 \ell_p^2} + \langle \hat{H}_\phi \rangle \right) \psi = -\frac{\ell_p^2}{2a} \langle \chi_k | \partial_a^2 | \chi_k \rangle \psi , \quad (2.3)$$

where Λ is the cosmological constant, and one for the matter part (see Eq. (14a) of Ref. [5]),

$$\left[\frac{\ell_p^2}{a} \frac{\partial_a \psi}{\psi} \partial_a + \left(\hat{H}_\phi - \langle \hat{H}_\phi \rangle \right) \right] | \chi_k \rangle = -\frac{\ell_p^2}{2a} \left(\partial_a^2 - \langle \chi_k | \partial_a^2 | \chi_k \rangle \right) | \chi_k \rangle , \quad (2.4)$$

with

$$H_\phi = \frac{1}{2} \left(\frac{\pi_\phi^2}{a^3} + a k^2 \phi^2 \right) . \quad (2.5)$$

It is important to remark that no approximation was invoked this far and Eqs. (2.3) and (2.4) are therefore equivalent to Eq. (2.1).

From now on, we require that a behaves like a (semi)classical quantity by assuming the validity of the WKB approximation for the gravitational wavefunction,

$$\psi \simeq \frac{1}{\sqrt{P_a}} e^{-i \int^a P_a(x) dx} , \quad (2.6)$$

where

$$P_a = -\ell_p^{-2} a \dot{a} \quad (2.7)$$

is the classical momentum associated to a . Again, after some manipulations and redefinitions, this yields the approximate matter equation (see Eq. (18) of Ref. [5])

$$\left[1 - \frac{3i \ell_p^2}{2a^3 \mathcal{H}} \left(1 + \frac{2\dot{\mathcal{H}}}{3\mathcal{H}^2} \right) \right] \left(i \partial_\tau - \hat{H}_\phi \right) | \chi_s \rangle \simeq \frac{\ell_p^2}{2a^3 \mathcal{H}} \Delta \hat{O} | \chi_s \rangle , \quad (2.8)$$

where $\mathcal{H} \equiv \dot{a}/a \equiv \partial_\tau a/a$ is the Hubble “parameter” and the cosmic time τ is introduced by means of the semiclassical (WKB) relation [2]

$$\ell_{\text{p}}^2 (\partial_a \log \psi) \partial_a \simeq i a \partial_\tau . \quad (2.9)$$

Note that the matter state appears in Eqs. (2.3) and (2.8) in different representations, namely

$$|\chi_{\text{s}}\rangle = \exp \left(-i \int^\tau \langle \hat{H}_\phi \rangle dt \right) |\chi_k\rangle , \quad (2.10)$$

but the expectation value of the matter Hamiltonian is the same for both,

$$\langle \hat{H}_\phi \rangle \equiv \langle \chi_k | \hat{H}_\phi | \chi_k \rangle = \langle \chi_{\text{s}} | \hat{H}_\phi | \chi_{\text{s}} \rangle . \quad (2.11)$$

The terms in the right hand sides of Eqs. (2.3) and (2.8) represent quantum gravitational fluctuations generated by the presence of matter. In particular,

$$\hat{O} = \frac{2i}{\mathcal{H}} \langle \hat{H}_\phi \rangle \partial_\tau + \frac{1}{\mathcal{H}} \partial_\tau^2 + 3i \left(1 + \frac{2\dot{\mathcal{H}}}{3\mathcal{H}^2} \right) \hat{H}_\phi \quad (2.12)$$

and $\Delta \hat{O} \equiv \hat{O} - \langle \chi_{\text{s}} | \hat{O} | \chi_{\text{s}} \rangle$.

In Ref. [6], we considered the above equations for the particular case of the spatially flat de Sitter universe,

$$a_{\text{dS}} = \alpha e^{\mathcal{H}_0 \tau} , \quad (2.13)$$

where $\mathcal{H} = \mathcal{H}_0 = \sqrt{\Lambda/3}$ and α are constants. We were then able to show that the quantum gravitational fluctuations in the matter equation (2.8) determined by the operator (2.12) become negligible and the scalar field appears asymptotically free at very early times, when $a_{\text{dS}} \sim k \ell_{\text{p}}$ (which was named regime I in Ref. [5]). In fact, for

$$y = \frac{\ell_{\text{p}}^2}{2\mathcal{H}a^3} \gg 1 . \quad (2.14)$$

it is possible to write the scalar field wavefunction as

$$|\chi_{\text{s}}\rangle \sim |\chi_{\text{asy}}\rangle = \sqrt{\mathcal{N}} \exp \left(-i \int \frac{3\ell_{\text{p}}^2 \kappa^2}{2a^3} dt \right) |\kappa\rangle , \quad (2.15)$$

where $|\kappa\rangle$ are orthogonal eigenstates of the scalar field momentum,

$$\frac{\hat{\pi}}{\sqrt{3}\ell_{\text{p}}} |\kappa\rangle = \kappa |\kappa\rangle , \quad (2.16)$$

and \mathcal{N} a normalization factor. The corresponding matter energy (after subtracting the usual zero-point contribution) is given by

$$\langle \chi_{\text{asy}} | \hat{H}_\phi | \chi_{\text{asy}} \rangle = \frac{3\ell_{\text{p}}^2 \kappa^2}{2a^3} \simeq \frac{k}{a} \langle n \rangle , \quad (2.17)$$

where $\langle n \rangle$ is the mean occupation number (in terms of the usual particle interpretation) for the mode k , which was estimated to be very close to zero [6]. The compatibility of these matter states with the de Sitter evolution was checked by showing that their energy (2.17) remains subleading with respect to the cosmological constant Λ and the corresponding quantum gravitational fluctuations also vanish in the semiclassical Friedmann equation (2.3).

The coupled dynamics of this matter-gravity system in the asymptotic regime was not fully exploited in Ref. [6] and we shall here complete its analysis.

3 The equation for gravity at very early times

We now want to study Eq. (2.3) for the gravitational degree of freedom in the very early universe, back to the limit of validity of the semiclassical approximation, that is for

$$a \sim a_0 \equiv a(\tau_0) = k \ell_{\text{p}} , \quad (3.1)$$

without assuming a specific time dependence for $a = a(\tau)$ ¹.

The expectation values in Eq. (2.3) are taken on the asymptotic matter states given in Eq. (2.15). From Eq. (2.10), we therefore find the asymptotic behaviour

$$|\chi_k\rangle \simeq \sqrt{\mathcal{N}} |\kappa\rangle = \sqrt{\mathcal{N} \frac{\sqrt{3} \ell_{\text{p}}}{2\pi}} e^{i\sqrt{3}\kappa \ell_{\text{p}} \phi} \quad (3.2)$$

which is clearly independent from a and leads to asymptotically vanishing gravitational perturbations,

$$\langle \chi_k | \partial_a^2 | \chi_k \rangle \simeq 0 . \quad (3.3)$$

In the asymptotic regime $y \gg 1$ (see Eq. (2.14)) and $a \sim k \ell_{\text{p}}$, the gravitational equation hence simplifies to

$$\left(\frac{\ell_{\text{p}}^2}{2a} \partial_a^2 + \frac{a^3 \Lambda}{6 \ell_{\text{p}}^2} + \langle \hat{H}_\phi \rangle \right) \psi \simeq 0 . \quad (3.4)$$

On substituting the WKB (semiclassical) form (2.6) for the wavefunction $\psi = \psi(a)$, one now gets the following evolution equation for the scale factor

$$\frac{\ell_{\text{p}}^2}{2a} \left[\frac{3}{4a^2} + \frac{\ddot{a}}{2a\dot{a}^2} + \frac{5\ddot{a}^2}{4\dot{a}^4} - \frac{\ddot{\ddot{a}}}{2\dot{a}^3} - \frac{a^2 \dot{a}^2}{\ell_{\text{p}}^4} \right] + \langle \hat{H}_\phi \rangle + \frac{\Lambda a^3}{6 \ell_{\text{p}}^2} \simeq 0 , \quad (3.5)$$

where $\langle \hat{H}_\phi \rangle$ is given in Eq. (2.17). The next step is to transform it into an equation for the Hubble parameter $\mathcal{H} = \mathcal{H}(\tau)$,

$$\mathcal{H}^6 - 2 \left(\frac{\ell_{\text{p}}^4}{a^6} + k \frac{\ell_{\text{p}}^2}{a^4} \langle n \rangle + \frac{\Lambda}{6} \right) \mathcal{H}^4 + \frac{\ell_{\text{p}}^4}{4a^6} (2\mathcal{H}\ddot{\mathcal{H}} - 6\mathcal{H}^2\dot{\mathcal{H}} - 5\dot{\mathcal{H}}^2) \simeq 0 . \quad (3.6)$$

It is worth noting that one can recover the de Sitter solution from the above equation for very large a . In fact, if we make the ansatz $\dot{\mathcal{H}} = 0$, Eq. (3.6) becomes

$$\mathcal{H}^2 = 2 \left(\frac{\ell_{\text{p}}^4}{a^6} + k \frac{\ell_{\text{p}}^2}{a^4} \langle n \rangle + \frac{\Lambda}{6} \right) , \quad (3.7)$$

and the limit $a \gg \ell_{\text{p}}$ yields the standard relation $\mathcal{H}^2 = \Lambda/3$. The second bracket in Eq. (3.6) therefore contains terms that qualify as *non-adiabatic*, since they would be neglected in the adiabatic approximation for the cosmic scale factor, that is for $|\ddot{\mathcal{H}}| \ll |\dot{\mathcal{H}}|^{3/2} \ll \mathcal{H}^3$ [7].

Turning again our attention to Eq. (3.6), we recall that in Ref. [6] we found that $\langle n \rangle \simeq 0$. We then see that the condition (2.14) makes the last two terms multiplying \mathcal{H}^4 asymptotically negligible with respect to the first one,

$$\frac{\ell_{\text{p}}^6}{a^6} \sim \frac{1}{k^6} \gg \frac{\langle n \rangle}{k^3} + \frac{1}{6} \ell_{\text{p}}^2 \Lambda \sim \ell_{\text{p}}^2 \mathcal{H}_0^2 . \quad (3.8)$$

¹In Ref. [6], the asymptotic regime was taken at $\tau = \tau_{\text{asy}} \rightarrow -\infty$, whereas in the present work we rescale the constant α in Eq. (2.13) so that $\tau_{\text{asy}} \rightarrow \tau_0 > 0$. This choice will appear necessary to ensure that the scale factor grows for increasing time (see Eqs. (3.13) and (3.14)).

Further, the first term in Eq. (3.6), which would dominate for large a , can also be neglected asymptotically (with respect to the second one). In fact, again on using Eq. (2.14), we obtain

$$a^6 \mathcal{H}^6 \sim k^6 \mathcal{H}_0^6 \ell_p^6 \ll \mathcal{H}_0^4 \ell_p^4 . \quad (3.9)$$

Finally, Eq. (3.6) in the asymptotic regime can be approximated as ²

$$2 \mathcal{H} \ddot{\mathcal{H}} - 6 \mathcal{H}^2 \dot{\mathcal{H}} - 5 \dot{\mathcal{H}}^2 - 8 \mathcal{H}^4 \simeq 0 . \quad (3.10)$$

This equation (like the complete Eq. (3.6)) does not admit a constant solution for \mathcal{H} . A solution can instead be found in the form of a power-law,

$$\mathcal{H} = \gamma \tau^\beta . \quad (3.11)$$

where γ and β are constants. Substituting into Eq. (3.10), we find $\beta = -1$ and a quartic equation for γ . Two of its four solutions coincide with $\gamma = 0$, that is $\mathcal{H} = 0$, which we discard, since it cannot be matched smoothly with the de Sitter evolution, and we are left with the solutions

$$\mathcal{H}_{1/4} = \frac{1}{4} \tau^{-1} \quad \text{and} \quad \mathcal{H}_{1/2} = \frac{1}{2} \tau^{-1} , \quad (3.12)$$

which yield the following forms for the scale factor

$$a_{1/4} = a_0 \left(\frac{\tau}{\tau_0} \right)^{1/4} , \quad (3.13)$$

and

$$a_{1/2} = a_0 \left(\frac{\tau}{\tau_0} \right)^{1/2} . \quad (3.14)$$

These expressions are, of course, understood to apply in the asymptotic regime only, that is for $\tau \simeq \tau_0$, with $a_0 = k \ell_p$. By matching $a_{\text{dS}}(\tau_0) \simeq a_0$ and ³

$$\mathcal{H}_{1/4}(\tau_0) \simeq \mathcal{H}_{1/2}(\tau_0) \simeq \tau_0^{-1} \simeq \mathcal{H}_0 , \quad (3.15)$$

one then finds $\alpha \sim a_0$. Moreover, at the time $\tau = \tau_0$, the requirements (3.8) and (3.9) both reduce to the same UV cut-off

$$k^3 \ll \frac{1}{\mathcal{H}_0 \ell_p} , \quad (3.16)$$

which is just the defining condition (2.14) for $a = a_0$.

It is important at this point to make sure that the above solutions (3.13) and (3.14) are compatible with all the approximations employed. In particular, we know that the leading order WKB approximation (2.6) is reliable if (see, for example, Ref. [14])

$$\left| \frac{dP_a}{da} \right| \ll P_a^2 . \quad (3.17)$$

²Similar equations arise in models containing both local quadratic corrections and (in contrast to Refs. [8, 9, 10]) the conformal anomaly terms. Their solutions near the cosmological singularity were studied in Ref. [11], while solutions with a bounded curvature and quasi-de Sitter behaviour were found in Ref. [12].

³Had we assumed $\tau_0 < 0$, the Hubble parameters (3.12) would be negative and this matching impossible. The solutions (3.13) and (3.14) for $\tau_0 < 0$ might however represent the collapse before a bounce [7] of the form considered, for example, in pre-Big Bang cosmology [13]. In this context, the case $\mathcal{H} = 0$ could also be accepted and the asymptotic regime interpreted as the bounce itself.

Upon substituting $a = a_{1/2}$, one finds that the corresponding momentum,

$$P_a = -\ell_p^{-2} a_0^2 \mathcal{H}_0 , \quad (3.18)$$

is constant and the condition (3.17) is thus trivially satisfied. However, for $a = a_{1/4}$,

$$P_a = -\frac{a_0^4 \mathcal{H}_0}{\ell_p^2 a_{1/4}} \quad (3.19)$$

and the condition (3.17) becomes

$$y_{1/4} \ll 1 , \quad (3.20)$$

in clear violation of Eq. (2.14). This already casts serious doubts about the validity of the solution (3.13), but we shall further show its inconsistency in the next Section.

Let us end this Section by noting that the solution $a_{1/2}$ coincides with the evolution of a classical universe filled with radiation, although the matter contribution was shown to be negligible and the radiation-like behaviour is of pure gravitational origin. Finally, the classical analogue to the solution $a_{1/4}$ would be a perfect fluid with equation of state parameter equal to $5/3$.

4 Reconsidering the matter equation

The results presented in the previous Section assume the asymptotic matter states obtained in Ref. [6] for the de Sitter evolution, but lead to different expressions for the scale factor. It is therefore necessary to check that the new solutions (3.13) and (3.14) are consistent with the chosen matter states.

4.1 Solution $a \propto \tau^{1/4}$

For the solution (3.13), one can easily see that

$$y_{1/4} \simeq \frac{\ell_p^2 a_{1/4}}{2 a_0^4 \mathcal{H}_0} , \quad (4.1)$$

which decreases for decreasing a , contrary to what happens in de Sitter, and there is no proper regime I. For $a_{1/4} \simeq a_0$, the right hand side of the matter equation (2.8) still vanishes on the asymptotic states (2.15) and the left hand side reduces to

$$\left(i \partial_\tau - \hat{H}_\phi \right) |\chi_s\rangle \simeq \left[i \partial_y - \left(\frac{2}{\ell_p^2} \hat{\pi}^2 + k^2 \frac{a_0^{16} y^4}{8 \tau_0^4 \ell_p^{10}} \hat{\phi}^2 \right) \right] |\chi_s\rangle \simeq 0 . \quad (4.2)$$

The kinetic term now dominates when y is small (as also required by the WKB approximation, see Eq. (3.20)), rather than large, and the asymptotic matter states found in Ref. [6] seem to remain valid solutions. However, the condition $y_{1/4} \ll 1$ estimated at the asymptotic time $\tau = \tau_0$ gives

$$k \gg (2 \mathcal{H}_0 \ell_p)^{-1/3} , \quad (4.3)$$

which violates the condition (3.16). Since this implies that the conditions (3.8) and (3.9) are also violated, this case cannot be accepted as a valid description of the coupled dynamics.

4.2 Solution $a \propto \tau^{1/2}$

For the solution (3.14), one analogously finds that

$$y_{1/2} \simeq \frac{\ell_p^2}{2 a_0^2 \mathcal{H}_0 a_{1/2}} , \quad (4.4)$$

which increases for decreasing a , like in de Sitter. It then follows that, contrary to the previous case, in the regime $y_{1/2} \gg 1$ (equivalent to Eq. (3.16) for $\tau = \tau_0$) the approximations required for the validity of Eq. (3.10) now hold. At the same time, since $\Delta\hat{O}|\chi_{\text{asy}}\rangle \simeq 0$, the states (2.15) remain asymptotic solutions to Eq. (2.8). Let us also note that the mode k lies inside the Hubble radius at the asymptotic time τ_0 if

$$\frac{a}{k} \mathcal{H} \sim \mathcal{H}_0 \ell_p \ll 1 . \quad (4.5)$$

If this condition is also satisfied, the UV cutoff on k implied by Eq. (3.16) is large and the range of allowed momenta sizable.

We can then conclude that only the solution $a_{1/2}$ in Eq. (3.14) can be accepted and is compatible with the asymptotic matter states obtained in Ref. [6]. Of course, all the disclaimers discussed in that paper about the existence of regime I and the condition (3.16) still apply. It is in particular possible that the universe was born with an initial scale factor $a \gg a_0$ and the asymptotic regime we have found never occurred [15].

5 Summary and conclusions

We have investigated the expansion of the scale factor in the very early universe, around the onset of inflation, by studying the gravitational equation obtained from the Born-Oppenheimer reduction of the Wheeler-DeWitt equation for one mode of a minimally coupled scalar field in the de Sitter space-time. We have found that quantum gravitational fluctuations due to the presence of matter asymptotically vanish going “backward in time” toward the limit of applicability of the semiclassical approximation, both in the equation for gravity and that for matter. The gravitational equation can then be solved and there emerges a phase during which the cosmological constant and the scalar field energy are negligible with respect to purely gravitational terms: in a sense, gravity seems to determine its own (initial) evolution.

In particular, we have uniquely obtained a “spontaneous” power-law evolution for the scale factor which reproduces the usual behaviour of a radiation-filled universe. We wish to remark once more that, similarly to the corrections near the FRW singularities studied in Ref. [7], this radiation-like evolution is driven by terms in the WKB form of the gravitational wavefunction and is thus essentially due to the initial non-classicality of the space-time. Inflation starts at a later stage, when the universe becomes purely classical, and is driven by a pre-existing cosmological constant which was ineffective during the semiclassical phase. It is also interesting that a similar behaviour near the singularity is known to be generic for the Einstein gravity with local quadratic corrections [8, 9] and was also analysed long ago in Ref. [10] and more recently, for example, in Refs. [16, 17].

Let us conclude by mentioning that our equations also suggest a different scenario in which the asymptotic regime corresponds to a “bounce” of the cosmic scale factor (that is $\mathcal{H} = 0$ or $\mathcal{H} < 0$) of the form employed in pre-Big Bang cosmology [13]. This possibility was however not considered in details here, since it requires that the universe went through an intermediate stage prior to the de Sitter expansion for which we have no analytical description.

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